DAMPING OF CAPILLARY-GRAVITATIONAL WAVES IN A VISCOUS FLUID OF FINITE DEPTH BY SURFACTANTS

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The damping of waves on the surface of a viscous fluid covered with a surfactant film has been studied by several authors. Methematical investigation of this phenomenon involves considerable difficulties. Levich in [1] proposed a hydrodynamic theory of wave damping and solved the problem of wave damping on the surface of an infinitely deep fluid. However, the exact solution of the problem for a fluid of finite depth is quite cumbersome and difficult of surveillance, necessitating the construction of simple approximate solutions. Moiseev [2 and 3] suggested a method for constructing asymptotic solutions for the case where the Reynolds number is large. In the author's studies [4 and 5] this method was used to solve a number of problems concerning a clean free surface. The present paper contains the solution of the plane linear problem on the damping of waves on a viscous fluid of finite depth whose free surface is covered with a surfactant. The formulation of the problem was communicated to the author by Levich.

Let the x-axis of the coordinate system Oxy coincide with the unperturbed surface of the fluid and let the y-axis be directed vertically upward. We introduce the dimensionless variables

$$\xi = \frac{x}{h}, \quad \eta = \frac{y}{h}, \quad \tau = \frac{t}{T}, \quad \mathbf{u} = \frac{T}{a}\mathbf{v}, \quad P = \frac{pT^2}{\rho a h}$$

Here h is the depth of the fluid, T is the characteristic time, a is the characteristic amplitude, V is the velocity, and p is the pressure. We reduce the equations of motion of the viscous fluid to the following form:

$$\frac{\partial \mathbf{u}}{\partial \tau} + \nabla\left(P + \frac{S}{F}\eta\right) = \frac{1}{R}\Delta \mathbf{u}, \quad \nabla \mathbf{u} = 0 \tag{1}$$

$$S = \frac{h}{a}, \quad \mathbf{v} = (v_1, v_2), \quad \mathbf{u} = (u_1, u_2), \quad R = \frac{h^2}{\nu T}, \quad F = \frac{(h/T)^2}{gh}$$

Here R is the Reynolds number, F is the Froude number, and the equation of the free surface $y = af(x, t)$ in dimensionless variables is of the form

$$\eta = \frac{1}{S}\zeta(\xi, \tau) \tag{2}$$

The condition of adhesion

$$u_1 = u_2 = 0 \quad \text{for } \eta = -1 \tag{3}$$

must be fulfilled at the bottom.

On the free surface the boundary conditions are of the form [1]

$$p_{nn} + p_\alpha = 0, \qquad p_{\gamma\gamma} + p_\gamma = 0 \qquad \text{for } \eta = 0 \tag{4}$$
$$p_\alpha = -\alpha(\Gamma)\, a \frac{\partial^2 f}{\partial x^2}, \qquad p_\gamma = \frac{\partial \alpha}{\partial \Gamma} \frac{\partial \Gamma}{\partial x}$$

Here p_{nn} is the normal component of the tension vector; $p_{\gamma\gamma}$ is the tangential component of the tension vector; p_α is the capillary pressure; p_γ is the tension associated with the tangential force due to the presence of the surfactant film on the free surface; $\Gamma = \Gamma(x)$ is the surfactant concentration; $\alpha(\Gamma)$ is the variable coefficient of surface tension.

To these conditions we must also add the kinetic condition

$$a \frac{\partial f}{\partial t} = v_2$$

Conditions (4) in dimensional parameters are of the form

$$-p + 2\mu v_{2y} = \alpha(\Gamma)\, a \frac{\partial^2 f}{\partial x^2}, \qquad \mu\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right) = \frac{\partial \alpha}{\partial \Gamma} \frac{\partial \Gamma}{\partial x} \tag{5}$$

With large Reynolds numbers it is also interesting to consider those surfactants which have the maximum effect on damping. In this limiting case, the surfactant film represents an incompressible plate which oscillates vertically. We shall assume also that the coefficient of surface tension $\alpha(\Gamma)$ is weakly dependent on the surfactant concentration and will therefore consider it constant. As was shown by Levich [1], in the case described boundary condition (5) on the free surface must be replaced by the condition

$$v_1(x, 0, t) = 0 \tag{6}$$

In dimensionless variables conditions (5) and (6) become

$$-P + \frac{2}{R} u_{2\eta} = K \frac{\partial^2 \zeta}{\partial \xi^2} \qquad \text{for } \eta = 0 \left(K = \frac{\alpha T^2}{\rho h^3}\right) \tag{7}$$

$$u_1 = 0 \qquad \text{for } \eta = 0 \tag{8}$$

We represent the velocity vector $\mathbf{u}$ as follows:

$$\mathbf{u} = \nabla^* \varphi + \nabla^* \psi \qquad (\nabla^* \psi = (\psi_\eta, -\varphi_\xi)) \tag{9}$$

Then, as we know [2 and 6], equations of motion (2) and the boundary conditions can be written as

$$\Delta\varphi = 0, \qquad \psi_\tau = \frac{1}{R} \Delta\psi, \qquad \varphi_\tau + \frac{S}{F}\eta + P = 0 \tag{10}$$

$$\varphi_\xi + \psi_\eta = 0, \qquad \varphi_\eta - \psi_\xi = 0 \qquad (\eta = -1) \tag{11}$$

$$\varphi_{\tau\tau} + \frac{1}{F}(\varphi_\eta - \psi_\xi) = \frac{2}{R} \frac{\partial}{\partial \tau} (\psi_{\xi\eta} - \varphi_{\eta\eta}) + K \frac{\partial^3 \zeta}{\partial \tau\, \partial \xi^2} \tag{12}$$
$$\varphi_\xi + \psi_\eta = 0 \qquad (\eta = 0)$$

The first condition in (12) was obtained with the aid of (7), the last relation in (10), and the kinematic condition.

We shall now turn to the problem of free oscillations attempting to solve problem (10) to (12) in the form

$$\varphi = \Phi(\xi, \eta)\, e^{\sigma\tau}, \quad \psi = \Psi(\xi, \eta)\, e^{\sigma\tau} \tag{13}$$

Where σ is some (unknown) complex number which determines the damping decrement and the oscillation frequency. In this case the problem becomes

$$\Delta\Phi = 0, \qquad \sigma\Psi = R^{-1} \Delta\Psi \tag{14}$$

$$\Phi_\xi + \Psi_\eta = 0, \qquad \Phi_\eta - \Psi_\xi = 0 \qquad \text{for } \eta = -1 \tag{15}$$

$$\sigma^2\Phi + F^{-1}(\Phi_\eta - \Psi_\xi) = 2\sigma R^{-1}(\Psi_{\xi\eta} - \Phi_{\eta\eta}) + K(\Phi_{\eta\xi\xi} - \Psi_{\xi s}) \tag{16}$$
$$\Phi_\xi + \Psi_\eta = 0 \qquad \text{for } \eta = 0$$

An approximate solution of the problem (14) to (16) can be obtained with the aid of the asymptotic method proposed by Moiseev [2 and 3]. The author employed this method previously [4 and 5] in constructing the asymptotic solutions of several problems on the oscillation of a viscous fluid with a clean free surface. The concept on which the method is based is as follows. For large R, a small parameter appears in front of the higher-order derivatives in Equation (14). If this parameter is set equal to zero, then problem (14) to (16) degenerates into the corresponding problem about waves on the surface of an ideal fluid ($\Psi \equiv 0$ and conditions (15) and (16) do not apply). If this parameter differes from zero and is sufficiently small, then the function Ψ is a function of the boundary layer type [7].

The function Ψ compensates the inconsistency of boundary conditions (15) and (16), since these conditions cannot be satisfied with the aid of function Φ alone. In accordance with the indicated character of the function Ψ, the latter is replaced for consideration by the function Ψ_1 and Ψ_2 which satisfy Equation (14); Ψ_1 compensates the inconsistency of boundary conditions (15), while Ψ_2 compensates that of (16). In addition, the functions Ψ_1' and Ψ_2 must vanish far away from the corresponding boundary, i.e. $\lim \Psi_1 = 0$ as $\eta \to \infty$, $\lim \Psi_2$ as $\eta \to -\infty$.

Asymptotic representations for the function Ψ_1 and Ψ_2 can now be found by expanding the required functions into asymptotic series in powers of $1/\sqrt{R}$. The first terms of such expansions are of physical interest. In particular, we find that the function Ψ_1 (as well as the function Ψ_2) can be found from the ordinary differential equation

$$\sigma\Psi_1 = \frac{1}{R}\,\Psi_{1\eta\eta}$$

to within $O(1/R^{3/2})$.

Without considering the details involved in the construction of the asymptotic solution (see [4 and 5]), let us cite the results of the computations to within $O(1/R)$ (the form of the solution corresponds to progressive waves)

$$\Phi = A\left\{\cosh[\omega(\eta+1)] + \frac{\omega}{\sqrt{\sigma R}}\sinh[\omega(\eta+1)]\right\}e^{-i\omega\xi} \qquad \left(\omega = kh = \frac{2\pi h}{\lambda}\right) \quad (17)$$

$$\Psi_1 = -\frac{i\omega A}{\sqrt{\sigma R}}\,e^{\sqrt{\sigma R}(\eta+1)-i\omega\xi}, \qquad \Psi_2 = -\frac{i\omega A\cosh\omega}{\sqrt{\sigma R}}\,e^{-\sqrt{\sigma R}\eta-i\omega\xi} \qquad (\mathrm{Re}\sqrt{\sigma}<0)$$

We note that in contrast to the case where the free surface is clean, the function Ψ_2 near the free surface has the same order $1/\sqrt{R}$ as the function Ψ_1 near the bottom. Substituting the expressions for Φ_1 and Φ_2 from (17) into (16), we obtain the approximate equation for determining σ

$$\sigma^2\cosh\omega + \omega\sinh\omega\left(\frac{1}{F} + K\omega^2\right) + \frac{\omega}{\sqrt{\sigma R}}\left[\sigma^2\sinh\omega + 2\omega\cosh\omega\left(\frac{1}{F} + K\omega^2\right)\right] = 0 \quad (18)$$

This equation is not valid for all dimensionless numbers ω, since for excessively small ω the Reynolds number no longer can be considered large, while for overly large ω (short waves) the linear theory is no longer applicable, and, in addition, the third term of Equation (18) is no longer small in comparison with the first two. However the interval of variation of ω is sufficiently broad (see [5]). If ω lies in the indicated interval, then, by virtue of Rouché's theórem, Equation (18) has two roots, i.e. exactly the same number as in the case of an ideal fluid. Seeking these roots in the form

$$\sigma = \sigma_0 + \frac{\sigma_1}{\sqrt{R}} + \frac{\sigma_2}{R} + \cdots \quad (19)$$

we find that to within $O(1/R)$

$$\sigma = \pm\sqrt{\omega\tanh\omega\,(F^{-1}+K\omega^2)}\,i - \frac{(1\pm i)\,\omega\sqrt[4]{\omega\tanh\omega\,(F^{-1}+K\omega^2)}}{\sqrt{2R}\,\sinh 2\omega}\,(1+\cosh^2\omega)$$

From this expression it follows that the oscillation frequency δ and the damping decrement β are given by Expressions

$$\delta = \sqrt{\omega \tanh \omega\,(F^{-1} + K\omega^2)} - \frac{\omega \sqrt[4]{\omega \tanh \omega\,(F^{-1} + K\omega^2)}}{\sqrt{2R}\sinh 2\omega}(1 + \cosh^2 \omega) \tag{20}$$

$$\beta = -\frac{\omega \sqrt[4]{\omega \tanh \omega\,(F^{-1} + K\omega^2)}}{\sqrt{2R}\sinh 2\omega}(1 + \cosh^2 \omega) \tag{21}$$

For comparison, let us cite analogous expressions for the case of a clean surface (δ_0 is the oscillation frequency and β_0 is the damping decrement) [4]

$$\delta_0 = \sqrt{\omega \tanh \omega\,(F^{-1} + K_0\omega^2)} - \frac{\omega \sqrt[4]{\omega \tanh \omega\,(F^{-1} + K_0\omega^2)}}{\sqrt{2R}\sinh 2\omega} \tag{22}$$

$$\beta_0 = -\frac{\omega \sqrt[4]{\omega \tanh \omega\,(F^{-1} + K_0\omega^2)}}{\sqrt{2R}\sinh 2\omega} \tag{23}$$

where $K_0 = \alpha_0 T^2 / \rho h^3$ and α_0 is the coefficient of surface tension.

Considering Formulas (20) to (23), we arrive at the following conclusions. For long waves (ω sufficiently small), the damping decrement β is twice as large as β_0 in absolute value. For shorter waves this difference increases: as ω increases, $|\beta|$ increases while $|\beta_0|$ diminishes. In addition, both β and β_0 (in contrast to the case of an infinitely deep fluid) depend on the coefficient of surface tension. The oscillation frequencies δ and δ_0 are smaller than for an ideal fluid. The amount by which the oscillation frequency diminishes is in both cases (to within $O(1/R)$) equal to the absolute value of the damping decrement.

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BIBLIOGRAPHY

1. Levich, V.G., Fiziko-khimicheskaia gidrodinamika (Physico-Chemical Hydrodynamics). Fizmatgiz, Moscow, 1959.

2. Moiseev, N.N., O kraevykh zadachakh dlia linearizirovannykh uravnenii Nav'e-Stoksa v sluchae, kogda viazkost' mala (On boundary value problems for linearized Navier-Stokes equations in cases where the viscosity is low). Zh.vychisl.Mat.mat.Fiz., Vol.1, № 3, pp.548-550, 1961.

3. Bagaeva, N.Ia. and Moiseev, N.N., Tri zadachi o kolebanii viazkoi zhidkosti (Three problems on the oscillation of a viscous fluid). Zh.vychisl.Mat.mat.Fiz., Vol.4, № 2, pp.317-326, 1964.

4. Shmidt, A.G., Gravitatsionnye i kapilliarnye volny na poverkhnosti sharovogo sloia viazkoi gravitiruiushchei zhidkosti (Gravitational and capillary waves on the surface of a spherical layer of a viscous gravitating fluid). Zh.vychisl.Mat.mat.Fiz., Vol.4, № 1, pp.183-189, 1964.

5. Shmidt, A.G., Kolebaniia viazkoi zhidkosti konechnoi glubiny, vyzvannye nachal'nym smeshcheniem ee svobodnoi poverkhnosti (Oscillations of a viscous fluid of finite depth occasioned by an initial displacement of its free surface). Zh.vychisl.Mat.mat.Fiz., Vol.5, № 2, pp.287-297, 1965.

6. Sretenskii, L.N., O volnakh na poverkhnosti viazkoi zhidkosti (on waves on the surface of a viscous fluid). Trudy tsen.aero-gidrodin.Inst., № 541, 1941.

7. Vishik, M.I. and Liusternik, L.A., Reguliarnoe vyrozhdenie i pogranichnyi sloi dlia lineinykh differentsial'nykh uravnenii s malym parametrom (Regular degeneration and the boundary layer for linear differential equations with a small parameter). Usp.mat.Nauk, Vol.12, № 5(77), pp.3-120, 1957.

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